# Signals and Systems

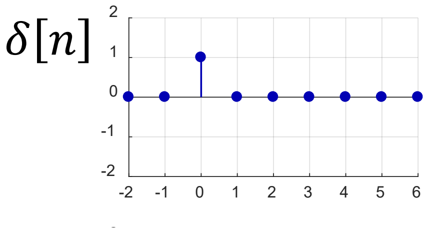
## Convolution

# The Unit Impulse

A discrete **impulse** signal is composed of all zero samples, except a single non-zero sample.

A discrete **unit impulse** signal is composed of all zero samples except for a single sample at the origin which has a value of one.

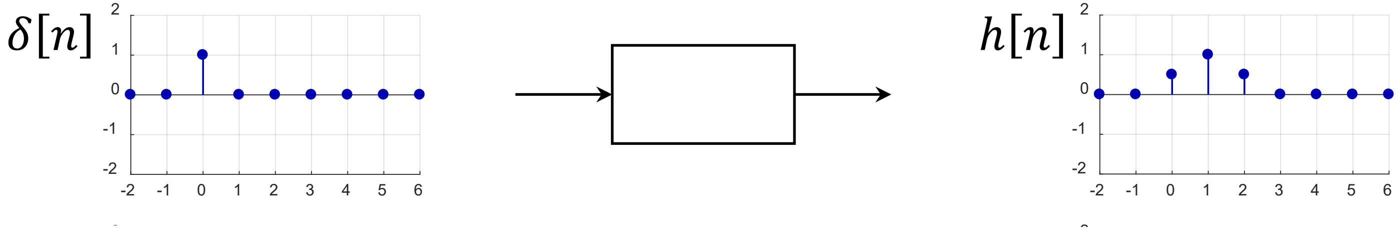
The discrete unit impulse signal is usually denoted by $\delta[n]$ and is therefore commonly called the discrete **delta function**.



# The Impulse Response of a System

The **impulse response** of a system is the output of the system when a unit impulse is the input.

The impulse response is usually denoted by $h[n]$.



# Discrete Convolution

The discrete convolution operation is formally defined by the following equation which is called the **convolution sum**.

$$y[n] = \sum_{m=0}^{M-1} f[m]g[n-m]$$

The convolution sum is normally written using the following shorthand notation:

$$y[n] = f[n] * g[n]$$

# Discrete Convolution

In typical applications for discrete convolution $f[n]$ usually has many fewer samples than $g[n]$.

If $f[n]$ consists of 3 samples, the equations for $y[n]$ can be written as:

$$y[0] = g[-2]f[2] + g[-1]f[1] + g[0]f[0]$$

$$y[1] = g[-1]f[2] + g[0]f[0] + g[1]f[0]$$

$$y[2] = g[0]f[2] + g[1]f[1] + g[2]f[0]$$

$$y[3] = g[1]f[2] + g[2]f[1] + g[3]f[0]$$

$$y[4] = g[2]f[2] + g[3]f[1] + g[4]f[0]$$

$$y[5] = g[3]f[2] + g[4]f[1] + g[5]f[0]$$

# Discrete Convolution

Notice that these equations represent the process of:

1. Multiplying the input samples by the left-right flipped impulse response.
2. Summing the result to give the output sample.
3. Shifting the input samples from right to left for the next output sample.

$$y[0] = g[-2]f[2] + g[-1]f[1] + g[0]f[0]$$

$$y[1] = g[-1]f[2] + g[0]f[0] + g[1]f[0]$$

$$y[2] = g[0]f[2] + g[1]f[1] + g[2]f[0]$$

$$y[3] = g[1]f[2] + g[2]f[1] + g[3]f[0]$$

$$y[4] = g[2]f[2] + g[3]f[1] + g[4]f[0]$$

$$y[5] = g[3]f[2] + g[4]f[1] + g[5]f[0]$$

# Convolution Machine

The convolution machine is a graphical representation of this process:

# Convolution End Effects

Notice that, when the convolution output is $y[0]$ the required input is samples: $x[-3]$, $x[-2]$, $x[-1]$, and $x[0]$.

The problem is, three of these samples: $x[-3]$, $x[-2]$, and $x[-1]$ do not exist!

This same dilemma arises at the end of the signal, where the convolution requires samples to the right of the defined input signal.

One way to handle this problem is by adding zero samples to the ends of the input signal.

This is called **padding** the signal with zeros.

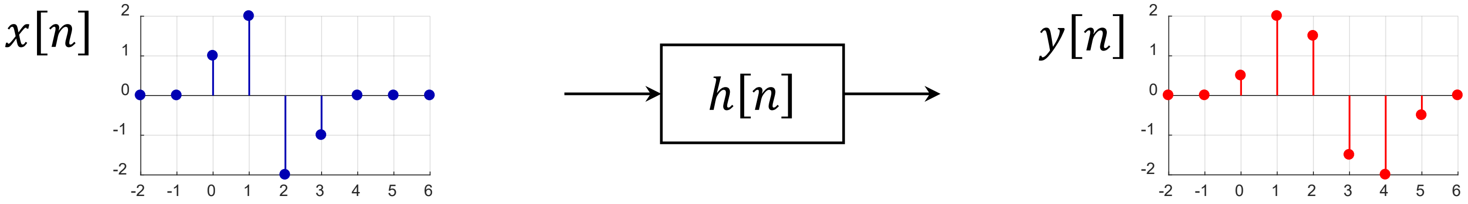
Since these zero values are eliminated during the multiplication, the result is mathematically the same as *ignoring* the non-existent inputs.

# The Importance of Discrete Convolution

Discrete convolution is an important operation in digital signal processing.

The output signal of any linear shift-invariant system is equal to **the convolution of the system's impulse response with the input signal** of the system:

$$y[n] = h[n] * x[n]$$



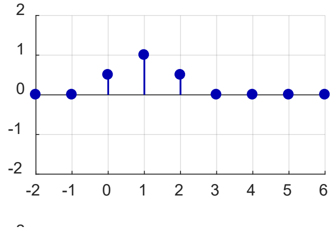
# Proof of $y[n] = h[n] * x[n]$

*For ease of notation, in the following proof, brackets are replaced with subscripts.*

Suppose that the impulse response of a linear shift-invariant system is given by:

$$h_n = h_0\delta_n + h_1\delta_{n-1} + h_2\delta_{n-2}$$

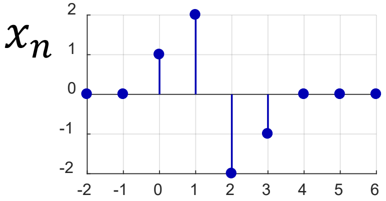
$h_n$

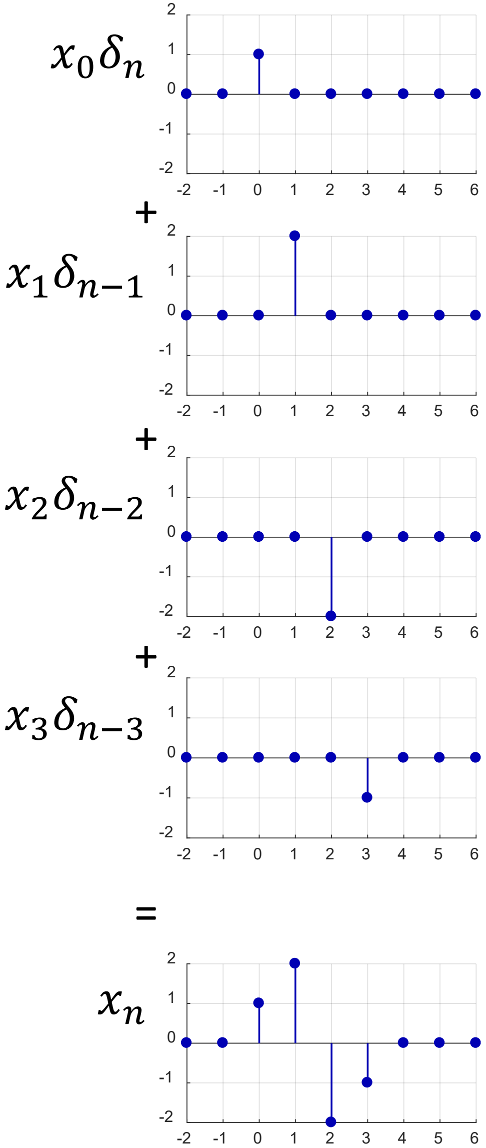


# Proof of $y[n] = h[n] * x[n]$

Then let the input to the system $x_n$ be represented by a sum of scaled and shifted unit impulses:

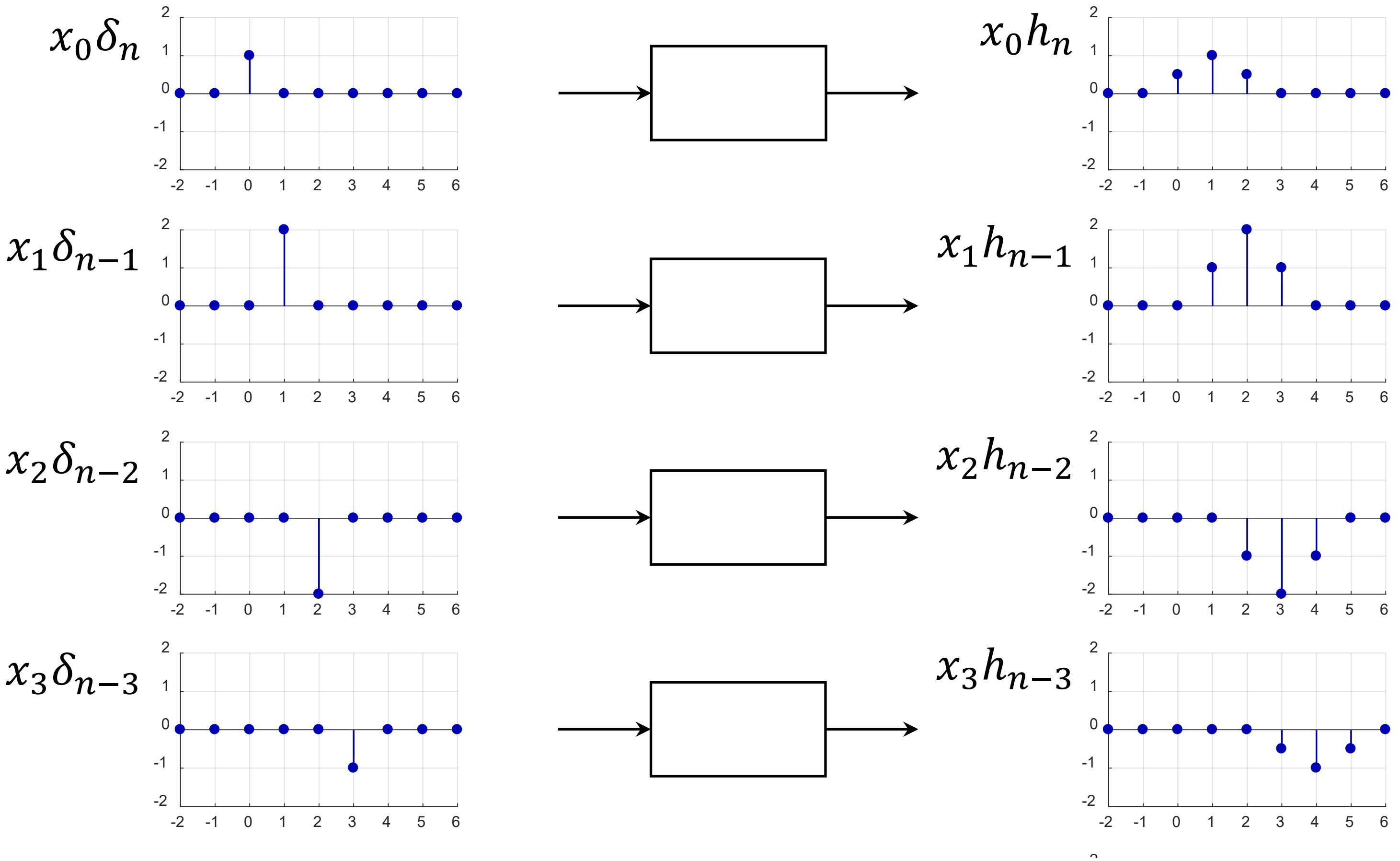
$$x_n = x_0\delta_n + x_1\delta_{n-1} + x_2\delta_{n-2} + x_3\delta_{n-3}$$



# Proof of $y[n] = h[n] * x[n]$

For a linear shift-invariant system, a scaled and shifted impulse on the input will produce a scaled and shifted impulse response on the output.



# Proof of $y[n] = h[n] * x[n]$

where the shifted impulse responses are given by:

$$h_{n-1} = h_0\delta_{n-1} + h_1\delta_{n-2} + h_2\delta_{n-3}$$

$$h_{n-2} = h_0\delta_{n-2} + h_1\delta_{n-3} + h_2\delta_{n-4}$$

$$h_{n-3} = h_0\delta_{n-3} + h_1\delta_{n-4} + h_2\delta_{n-5}$$

If the input to the linear shift-invariant system is the *sum* of scaled and shifted unit impulses, the output will be the *sum* of scaled and shifted impulse responses:

$$y_n = x_0 h_n + x_1 h_{n-1} + x_2 h_{n-2} + x_3 h_{n-3}$$

# Proof of $y[n] = h[n] * x[n]$

This equation for $y_n$ can be expanded to give:

$$y_n = x_0(h_0\delta_n + h_1\delta_{n-1} + h_2\delta_{n-2}) + x_1(h_0\delta_{n-1} + h_1\delta_{n-2} + h_2\delta_{n-3})$$
$$+ x_2(h_0\delta_{n-2} + h_1\delta_{n-3} + h_2\delta_{n-4}) + x_3(h_0\delta_{n-3} + h_1\delta_{n-4} + h_2\delta_{n-5})$$

Then rearranging to group like delta function terms gives:

$$y_n = (x_0h_0)\delta_n + (x_0h_1 + x_1h_0)\delta_{n-1} + (x_0h_2 + x_1h_1 + x_2h_0)\delta_{n-2}$$
$$+(x_1h_2 + x_2h_1 + x_3h_0)\delta_{n-3} + (x_2h_2 + x_3h_1)\delta_{n-4} + (x_3h_2)\delta_{n-5}$$

# Proof of $y[n] = h[n] * x[n]$

which gives the following equations for the individual samples of $y_n$:

$$y_0 = x_0 h_0$$

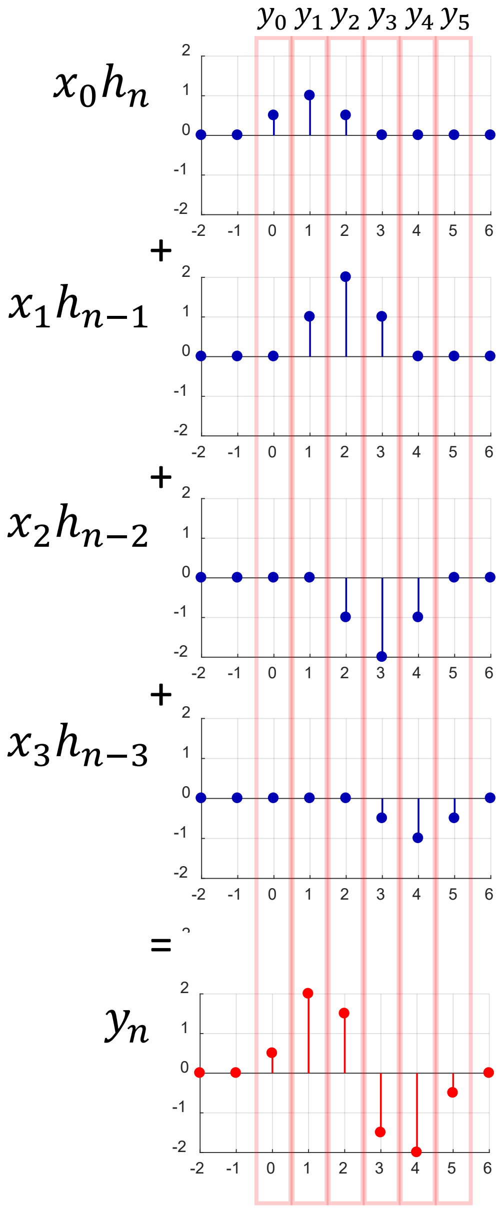
$$y_1 = x_0 h_1 + x_1 h_0$$

$$y_2 = x_0 h_2 + x_1 h_1 + x_2 h_0$$

$$y_3 = x_1 h_2 + x_2 h_1 + x_3 h_0$$

$$y_4 = x_2 h_2 + x_3 h_1$$

$$y_5 = x_3 h_2$$



# Proof of $y[n] = h[n] * x[n]$

Rewrite the equations for the individual samples of $y_n$ as shown below

$$
\begin{aligned}
y_0 &= x_0 h_0 \\
y_1 &= x_0 h_1 + x_1 h_0 \\
y_2 &= x_0 h_2 + x_1 h_1 + x_2 h_0 \\
y_3 &= x_1 h_2 + x_2 h_1 + x_3 h_0 \\
y_4 &= x_2 h_2 + x_3 h_1 \\
y_5 &= x_3 h_2
\end{aligned}
$$

Notice that these equations represent the process of shifting the input samples from right to left, multiplying the samples by the flipped impulse response and summing the result for each shift to give the output sample.

This process is the same as the discrete convolution of the input signal and the impulse response.

# Proof of $y[n] = h[n] * x[n]$

So the general equation for the samples of $y_n$ can be written as:

$$y_n = \sum_{m=0}^{\mathrm{M}} h_m x_{n-m}$$

or

$$y_n = h_n * x_n$$

# Convolution Example: Low Pass Filter

# Convolution Example: High Pass Filter

# Correlation Example: Matched Filtering

# Correlation Example: Matched Filtering with Noise

# Further Reading

Smith, S. W. (1997). The scientist and engineer's guide to digital signal processing.

Chapter 6 – Convolution

Chapter 7 – Properties of Convolution